A priori estimate for a family of semi-linear elliptic equations with critical nonlinearity

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Abstract

We consider positive solutions of $\Delta u - \mu u + K u^{\frac{n+2}{n-2}} = 0$ on B_1 ($n \geq 5$) where μ and $K > 0$ are smooth functions on B_1. If K is very sub-harmonic at each critical point of K in $B_{2/3}$ and the maximum of u in $\bar{B}_{1/3}$ is comparable to its maximum over $\bar{B}_1$, then all positive solutions are uniformly bounded on $\bar{B}_{1/3}$. As an application, a priori estimate for solutions of equations defined on $\mathbb{S}^n$ is derived.

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1 Introduction

In this article we study the equation

$$\Delta u - \mu(x) u + K(x) u^{n^*} = 0, \quad u > 0, \quad u \in C^2(B_1), \quad n \geq 5, \tag{1.1}$$

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where B_1 is the unit ball centered at the origin, $n^* = \frac{n+2}{n-2}$ is the critical power in Sobolev embedding, μ is a C^1 function on B_1 and $K \in C^3(B_1)$ is a positive function. We shall derive a priori estimate under natural assumptions on K and μ.

Equation (1.1) has rich connections in physics and geometry. In particular, it is very closely related to the well known Yamabe equation, which has been extensively studied for decades. Many interesting features of the Yamabe equation are also reflected on this equation. When μ is a non positive constant, many results on the existence of solutions, multiplicity of solutions, a priori estimates, bifurcation phenomena, Harnack type inequalities, etc can be found in the literature. We refer the interested readers to [3][6][9][28][31] and the references therein. On the other hand, much less references can be found for the case $\mu > 0$. A recent paper of Lin-Prapapet [25] discussed the case $\mu = constant > 0$ and they pointed out that it is also interesting and important to study the following Harnack type inequality:

$$(\max_{\bar{B}_{1/3}} u)(\min_{\bar{B}_{2/3}} u) \le C. \tag{1.2}$$

In a slightly different setting, they derived this Harnack inquality for $3 \le n \le 6$ under some flatness assumptions of ∇K near its critical points. They also speculated that (1.2) should still hold for higher dimensions under similar assumptions by finer analysis.

The Harnack inequality (1.2) is an important estimate to understand the blowup phenomena of (1.1) and many related equations with the critical Sobolev nonlinearity. The very first discussions of this inequality can be found in [27] and [11]. With this Harnack type inequality, usually the blowup phenomena is greatly simplified and some energy estimates are implied. Moreover, some further results such as a priori estimate, precise description of the blowup bubbles, etc can be obtained.

In this article, we use a very different approach from Lin-Prajapat's to obtain a priori estimate for general $\mu \in C^1$ and $n \ge 5$. We shall also derive the Harnack type inequality as an intermediate step toward our result. Our idea stems from the author's joint work with Y. Y. Li [21][23] on the compactness of solutions of the Yamabe equation.

We assume the following on K and μ:

$$(K,\mu): \quad C_1^{-1} \le K(x) \le C_1, \quad x \in B_1, \quad \|K\|_{C^3(B_1)} \le C_1, \quad \|\mu\|_{C^1(B_1)} \le C_0.$$

In addition, we need the maximum of u in $B_{1/3}$ comparable to its maximum in B_1: There exists $C_2 > 0$ such that

$$\max_{\bar{B}_{1/3}} u \ge \frac{1}{C_2} \max_{\bar{B}_1} u. \tag{1.3}$$

The main result of the paper is

Theorem 1.1. *Given (K,μ) and (1.3) there exists $C_3(C_0,n,C_1,C_2)>0$ such that for each critical point x of K in $B_{\frac{2}{3}}$, if $\Delta K(x)>C_3$, any solution u of (1.1) satisfies*

$$\max_{\bar{B}_{1/3}} u \le C_4(C_0,n,C_1,C_2). \tag{1.4}$$

If μ is a positive constant, Lin-Prajapet proved in [25] without the assumption (1.3) that for $n=3$, if K is Hölder continuous with exponent $\theta\in(\frac{1}{2},1]$ then (1.2) holds. For $4\le n\le 6$, if $K\in C^1$ and in a neighborhood of each critical point x_0 of K,

$$c|x-x_0|^{\theta-1}\le|\nabla K(x)|\le C|x-x_0|^{\theta-1}$$

holds for $c,C>0$ and $\frac{n-2}{2}\le\theta\le n-2$, then (1.2) holds.

Even though (1.3) seems to be a strong restriction for locally defined equations, it can be removed when Theorem 1.1 is applied to globally defined equations. In this case Theorem 1.1 is particularly useful. For example, let w be a positive solution to

$$Lw-aw+Rw^{\frac{n+2}{n-2}}=0,\quad \text{on}\quad S^n \tag{1.5}$$

where $L=\Delta_{g_0}-\frac{n(n-2)}{4}$ is the conformal Laplacian operator of (S^n,g_0), a and R are positive smooth functions. By using the stereographic projection π from S^n to $\mathbb{R}^n$, we set $K(y)=R(\pi^{-1}(y))$ for $y\in\mathbb{R}^n$ and $\mu(y)=a(\pi^{-1}(y))$. Without loss of generality we assume the north pole is not a critical point of R. Similar to (K,μ) we assume

$$(K,\mu)_1: \quad C_5^{-1}\le K\le C_5,\quad \|K\|_{C^3(\mathbb{R}^n)}\le C_5,\quad \|\mu\|_{C^1(\mathbb{R}^n)}\le C_5.$$

Then we have

Corollary 1.1. *Given $n\ge 5$ and $(K,\mu)_1$, then there exists $C_6(C_5,n)>0$ such that for each critical point y of K, if $\Delta K(y)>C_6$, solution w of (1.5) satisfies*

$$C_7^{-1}\le w(x)\le C_7,\quad x\in\mathbb{S}^n$$

where $C_7(C_5,n)>0$ is independent of w.

Remark 1.1. *If $a\equiv 0$ in (1.5), then we only need to assume $\Delta K(y)>0$ on critical point y to get the same a priori estimate.*

The results in Corollary 1.1 and Remark 1.1 can be compared with closely related results in [10][13][19],[20], etc and the references therein. Comparing to these results, Corollary 1.1 gives the a priori estimate under a very short assumption on K.

The conclusion of Theorem 1.1 can also be compared with related results on the compactness of solutions of the Yamabe equation. It is proved in [21] and [26] that for the Yamabe equation, blowup point can not appear at places where the Weyl tensor is 0. The role of ΔK in Theorem 1.1 is similar to that of the Weyl tensor for the Yamabe equation. Please also see [4][5] [10][14][18][22][24] for related discussions.

In another work of the author [30], among other things the following result is essentially proved: For $\mu = 0$ and $n \geq 5$, suppose (1.3) holds and $\Delta K(x) > 0$ for all critical point x in $B_{2/3}$, then $\max_{\bar{B}_{1/3}} u \leq C$ for some $C > 0$.

Based on Theorem 1.1 and the results in [25][30] we propose the following two questions for the case $\mu = constant > 0$:

1. For $n = 4$, under the assumptions of Theorem 1.1, can one obtain the Harnack type inequality (1.2) or even stronger, the a priori estimate?

2. For $n \geq 5$ under the assumptions (K, μ) and (1.3), what is the smallest $C > 0$ so that Theorem 1.1 holds for $\Delta K > C$ at each critical point in $B_{2/3}$? We suspect that the best constant only comes from the Pohozaev Identity.

The idea of the proof of Theorem 1.1 is by an iterative use of the well known method of moving spheres (MMS). Some estimates established by Chen-Lin [12] are very crucial to our approach. The reason that we need to apply MMS many times is because we need to construct appropriate test functions for this well known method. The construction of test functions depends on the estimates of some error terms and is closely related to the spectrum of the linearized operator of the equation. At the beginning we only have crude estimates of these error terms. As a consequence we can only construct test functions (and apply MMS) on these small domains. However, MMS and Chen-Lin's estimate lead to better estimates of error terms, which make it possible to construct test functions on larger domains. After applying this procedure iteratively we obtain the desired estimates on the error terms and MMS can be applied on roughly the whole domain.

More specifically the outline of the proof is as follows. Suppose there is no uniform bound for a sequence of solutions u_i. Scale u_i appropriately so that the maximum of the re-scaled function v_i is comparable to 1. Then these functions are defined on very large balls,say, $B(0, \frac{1}{10}M_i^{\frac{2}{n-2}})$, where M_i is the maximum of u_i. The first step is to show that v_i is comparable to a standard bubble that takes 1 at 0

over the range $B(0, M_i^{\frac{2}{(n-2)^2}})$. The way to prove this step is by an easy application of MMS on this range. Then in step two we show that the difference between v_i and the standard bubble is of the order $O(M_i^{-\frac{2}{n-2}})$. The approach for this step is based on Chen-Lin's argument in [12]. The result in step two helps us to describe some error terms in a better way so that we can use MMS to prove, in step three, that v_i is comparable to the standard bubble over the range $B(0, M_i^{\frac{4}{(n-2)^2}})$. In this step we need to rewrite some major error terms into a product of spherical harmonics with radial functions. This decomposition allows us to find test functions of the same form. By using the Pohozaev identity in step four we obtain that $|\nabla K|$ at the blowup point must vanish at the order of $O(M_i^{-\frac{2}{n-2}})$. Then in step five we apply the Chen-Lin estimate again to show that the closeness between v_i and the standard bubble can be improved to $O(M_i^{-\frac{4}{n-2}})$. This is optimal for this closeness. This new estimate helps us again to describe some error terms in a better way so that we can apply MMS again in a bigger domain. In fact in step six we show that in almost the whole domain, v_i is comparable to the standard bubble. In this step, the largeness of ΔK at the blowup point is used. Finally in step seven we apply the Pohozaev identity over the whole domain, using symmetry and all previous estimates to get a contradiction.

The organization of this paper is as follows. In section two the proof of Theorem 1.1 is presented. The different steps of the proof are contained in different subsections. At the end of section two we use Theorem 1.1 to prove Corollary 1.1.

2 The proof of Theorem 1.1

In the proof of Theorem 1.1 we only consider the case $\mu = constant > 0$. In our argument, the difference between μ being a general C^1 function and a constant only produces minor terms in our estimate. In order not to make notations difficult, we leave the general case for the interested readers.

The proof is based on an assumption for contradiction. Suppose a sequence of functions u_i can be found to satisfy

$$\Delta u_i - \mu_i u_i + K_i(x) u_i^{n^*} = 0 \quad B_1$$

with K_i satisfying (K, μ) and $0 \le \mu_i \le C_0$ such that

$$u_i(\bar{x}_i) = \max_{\bar{B}_{1/3}} u_i \to \infty. \tag{2.1}$$

Then by a standard selection process we have $x_i \in B(\bar{x}_i, 1/10)$ such that x_i is a local maximum of u_i and $M_i = u_i(x_i)$ is comparable to the maximum of u_i in B_1. Moreover

$$v_i(y) = M_i^{-1} u_i(M_i^{-\frac{2}{n-2}} y + x_i)$$

tends in $C^2_{loc}(\mathbb{R}^n)$ norm to U which satisfies

$$\Delta U + \lim_{i\to\infty} K_i(x_i) U^{\frac{n+2}{n-2}} = 0, \quad \mathbb{R}^n.$$

and $U(0) = 1 = \max_{\mathbb{R}^n} U$. By the well known classification theorem of Caffarelli-Gidas-Spruck we have

$$U(y) = (1 + |y|^2)^{-\frac{n-2}{2}}.$$

Here we have assumed without loss of generality that

$$\lim_{i\to\infty} K_i(x_i) = n(n-2).$$

The standard selection process can be found in quite a few papers, for example [30]. Direct computation shows that the equations for v_i is

$$\Delta v_i - \mu_i M_i^{-\frac{4}{n-2}} v_i + K_i(M_i^{-\frac{2}{n-2}} \cdot + x_i) v_i^{n^*} = 0, \quad \Omega_i \tag{2.2}$$

where $\Omega_i := B(0, \frac{1}{10} M_i^{\frac{2}{n-2}})$. In the sequel unless we state otherwise a constant always depends on n, C_0, C_1, C_2.

2.1 Estimate of v_i over $B(0, \delta M_i^{\frac{2}{(n-2)^2}})$

In this subsection we establish the following estimate:

Proposition 2.1. *For any $\epsilon > 0$ there exists $\delta_0(\epsilon) > 0$ such that for all large i*

$$\min_{|y|=r} v_i(y) \le (1+\epsilon) r^{2-n}, \quad \forall r \in (0, \delta_0 M_i^{\frac{2}{(n-2)^2}}).$$

Proof of Proposition 2.1: The proof is by a contradiction. Suppose there exists $\epsilon_0 > 0$ and $r_i = \circ(1) M_i^{\frac{2}{(n-2)^2}}$ so that

$$\min_{|y|=r_i} v_i(y) \ge (1+\epsilon_0) r_i^{2-n}. \tag{2.3}$$

Note that by the convergence of v_i to U we certainly have $r_i \to \infty$. We shall use the moving sphere argument. Here we let $\Sigma_\lambda = B_{r_i} \setminus \bar{B}_\lambda$. The boundary condition for v_i on $|y| = r_i$ is (2.3). Let

$$v_i^\lambda(y) = (\frac{\lambda}{|y|})^{n-2} v_i(y^\lambda), \quad y^\lambda = \frac{\lambda^2 y}{|y|^2}$$

be the Kelvin transformation of v_i with respect to ∂B_λ. Note that in this article λ is always assumed to stay between two positive constants independent of i. The equation for v_i^λ is

$$\Delta v_i^\lambda - \mu_i M_i^{-\frac{4}{n-2}} (\frac{\lambda}{|y|})^4 v_i^\lambda + K_i(M_i^{-\frac{2}{n-2}} y^\lambda + x_i)(v_i^\lambda)^{n^*} = 0 \quad \Sigma_\lambda. \tag{2.4}$$

Set $w_\lambda = v_i - v_i^\lambda$ in Σ_λ. For simplicity we omit i in this notation. We shall apply the moving sphere argument to w_λ with a test function. The equation for w_λ is

$$T_\lambda w_\lambda = E_\lambda \quad \text{in} \quad \Sigma_\lambda. \tag{2.5}$$

where

$$T_\lambda := \Delta - \mu_i M_i^{-\frac{4}{n-2}} + n^* K_i(M_i^{-\frac{2}{n-2}} y + x_i) \xi_\lambda^{\frac{4}{n-2}},$$

ξ_λ is obtained from the mean value theorem:

$$\xi_\lambda^{\frac{4}{n-2}} = \int_0^1 (t v_i + (1-t) v_i^\lambda)^{\frac{4}{n-2}} dt,$$

$$E_\lambda = \mu_i M_i^{-\frac{4}{n-2}} v_i^\lambda (1 - (\frac{\lambda}{|y|})^4) + (K_i(M_i^{-\frac{2}{n-2}} y^\lambda + x_i) - K_i(M_i^{-\frac{2}{n-2}} y + x_i))(v_i^\lambda)^{n^*} \tag{2.6}$$

is the main error term.

For the moving sphere argument we shall find two constants λ_0 and λ_1, both independent of i such that $\lambda_0 \in (\frac{1}{2}, 1)$, $\lambda_1 \in (1, 2)$. We shall only consider $\lambda \in [\lambda_0, \lambda_1]$ for the moving sphere method. Test function h_λ, which depends on i ($\lambda \in [\lambda_0, \lambda_1]$), will be constructed to satisfy

$$h_\lambda|_{\partial B_\lambda} = 0 \tag{2.7}$$

$$h_\lambda = \circ(1) r^{2-n} \quad \text{in} \quad \Sigma_\lambda, \quad |\nabla h_\lambda| = \circ(1) \quad \text{in} \quad \Sigma_\lambda \cap B_R, \tag{2.8}$$

$$T_\lambda h_\lambda + E_\lambda \le 0, \quad \text{in} \quad O_\lambda := \{ y \in \Sigma_\lambda; \quad v_i(y) \le 2 v_i^\lambda(y) \quad \} \tag{2.9}$$

where R in (2.8) is any fixed large constant.

Once such a test function is constructed the moving sphere argument can be applied to get a contradiction to (2.3). In fact first we show that the moving sphere process can get started at λ_0:

$$w_{\lambda_0} + h_{\lambda_0} > 0 \quad \text{in} \quad \Sigma_{\lambda_0}. \tag{2.10}$$

To see this, first we state a property of the standard bubble U:

$$U(y) - U^{\lambda}(y) \sim (1 - \frac{\lambda}{|y|})(1-\lambda)|y|^{2-n}, \quad |y| > \lambda, \tag{2.11}$$

which implies that for any $\lambda_0 < 1$ and $|y| > \lambda_0$

$$w_{\lambda_0}(y) > C(|y| - \lambda_0)|y|^{1-n}, \quad \lambda_0 < |y| < \bar{R}$$

for all large i by the convergence of w_{λ_0} to $U - U^{\lambda_0}$ in $C^2_{loc}(\mathbb{R}^n)$. Here $\bar{R}$ is a large fixed number to be determined. By (2.7) (2.8) one sees easily that $w_{\lambda_0} + h_{\lambda_0} > 0$ over $B_{\bar{R}} \cap \Sigma_{\lambda_0}$. For $|y| > \bar{R}$ we observe that

$$v_i^{\lambda_0}(y) \le \lambda_0^{n-2}(1+\epsilon_1)|y|^{2-n}, \quad |y| \ge \bar{R}$$

where ϵ_1 is sufficiently small so that $\lambda_0^{n-2}(1+\epsilon_1) < 1 - 5\epsilon_1$. On the other hand, by the convergence of v_i to U we can make $\bar{R}$ large enough so that

$$v_i(y) \ge (1 - 2\epsilon_1)|y|^{2-n}, \quad |y| = \bar{R}.$$

On $\Sigma_{\lambda_0} \setminus \bar{B}_{\bar{R}}$ v_i satisfies

$$\Delta v_i - \mu_i M_i^{-\frac{4}{n-2}} v_i \le 0, \quad \Sigma_{\lambda_0} \setminus \bar{B}_{\bar{R}}.$$

Let

$$\tilde{O} := \{ \ y \in \Sigma_{\lambda_0} \setminus \bar{B}_{\bar{R}}; \quad v_i(y) \le 2|y|^{2-n} \ \}.$$

Clearly $w_{\lambda_0} + h_{\lambda_0} > 0$ in $\Sigma_{\lambda_0} \setminus (B_{\bar{R}} \cup \tilde{O})$. Let G be the Green's function of $-\Delta$ on $\Sigma_{\lambda_0} \setminus \bar{B}_{\bar{R}}$ with respect to the Dirichlet boundary condition, let

$$\phi(y) = \int_{\tilde{O}} G(y, \eta) \mu_i M_i^{-\frac{4}{n-2}} v_i(\eta) d\eta.$$

Then

$$-\Delta \phi = \mu_i M_i^{-\frac{4}{n-2}} v_i \quad \tilde{O}.$$

Elementary estimate gives

$$\phi(y) \le C(n)\mu_i M_i^{-\frac{4}{n-2}} |y|^{4-n}. \tag{2.12}$$

So

$$\phi(y) \le \epsilon_1 |y|^{2-n}. \tag{2.13}$$

Now we have

$$\Delta(v_i + \phi) \le 0 \quad \text{in} \quad \tilde{O}.$$

By maximum principle

$$v_i + \phi > (1 - 3\epsilon_1)|y|^{2-n} \quad \text{in} \quad \tilde{O}.$$

By (2.13)

$$v_i(y) > (1 - 4\epsilon_1)|y|^{2-n} \quad \text{in} \quad \tilde{O}.$$

Since $h_\lambda = \circ(1)|y|^{2-n}$ for $\lambda \in (1/2, 2)$, (2.10) is established.

Once the moving sphere process can get started at λ_0, let $\bar{\lambda}$ be the critical position where $w_\lambda + h_\lambda$ ceases to be positive in Σ_λ. But because of (2.7), (2.8) and (2.9), the moving sphere process can reach λ_1, i.e. $\bar{\lambda} \ge \lambda_1$. Note that $T_\lambda h_\lambda + E_\lambda$ only needs to be non positive in O_λ because by (2.8) $w_\lambda + h_\lambda > 0$ in $\Sigma_\lambda \setminus O_\lambda$. Since $\lambda_1 > 1$, by letting $i \to \infty$ we have

$$U(y) - U^{\lambda_1}(y) \le 0, \quad |y| \ge \lambda_1,$$

which is contradictory to (2.11).

Remark 2.1. *What is described above is a general procedure of the application of moving sphere method, which will be used a few times in the sequel. Even though Σ_λ and h_λ will be different in different contexts, the important thing is to construct h_λ that satisfies (2.7), (2.8) and (2.9). The way to start the moving sphere process and to apply the maximum principle to get a contradiction from the standard bubbles are just the same and will not be repeated.*

To construct h_λ in this subsection we use the following crude estimate of E_λ:

$$|E_\lambda| \le CM_i^{-\frac{4}{n-2}} r^{2-n} + CM_i^{-\frac{2}{n-2}} r^{-n-1} \le CM_i^{-\frac{2}{n-2}} r^{2-n}, \quad \Sigma_\lambda. \tag{2.14}$$

Note that we use r to represent $|y|$. The construction of h_λ in this subsection is not subtle with respect to λ, we just set $\lambda_0 = \frac{1}{2}$ and $\lambda_1 = 2$. We need the following non positive function: For $2 < \alpha < n$, let

$$f_{n,\alpha}(r) = -\frac{1}{(n-\alpha)(2-\alpha)}(r^{2-\alpha} - \lambda^{2-\alpha}) - \frac{\lambda^{n-\alpha}}{(n-\alpha)(n-2)}(r^{2-n} - \lambda^{2-n}).$$

By direct computation one verifies that

$$\begin{cases} \Delta f_{n,\alpha}(r) = f''_{n,\alpha}(r) + \frac{n-1}{r} f'_{n,\alpha}(r) = -r^{-\alpha}, \quad r \geq \lambda, \\ f_{n,\alpha}(\lambda) = f'_{n,\alpha}(\lambda) = 0, \\ 0 \leq -f_{n,\alpha}(r) \leq C(n,\alpha). \end{cases} \tag{2.15}$$

This function is mainly used to control minor terms. We define h_λ as

$$h_\lambda = Q M_i^{-\frac{2}{n-2}} f_{n,n-2}(\frac{r}{\lambda})$$

where Q is a large number to be determined. Here we see that by (2.15), (2.7) and (2.8) hold. Also $h_\lambda \leq 0$ in Σ_λ. Now we verify (2.9). First by choosing Q large enough we have

$$\Delta h_\lambda + E_\lambda \leq -\frac{Q}{2} M_i^{-\frac{2}{n-2}} r^{2-n}, \quad \Sigma_\lambda.$$

Since h_λ is non positive, the term $n^* K_i(M_i^{-\frac{2}{n-2}} y + x_i)\xi_\lambda^{\frac{4}{n-2}} h_\lambda$ is also non positive. The only thing we need to verify is

$$-\mu_i M_i^{-\frac{4}{n-2}} h_\lambda \leq \frac{Q}{4} M_i^{-\frac{2}{n-2}} r^{2-n} \quad \text{in} \quad \Sigma_\lambda.$$

By direct computation this holds. Proposition 2.1 is established. $\square$

Next we establish the closeness between v_i and U_i, which satisfies:

$$\Delta U_i + K_i(x_i) U_i^{n^*} = 0, \quad \mathbb{R}^n. \quad U_i(0) = 1 = \max_{\mathbb{R}^n} U_i.$$

Proposition 2.2. *There exist $\delta_1 > 0$ and $C > 0$ such that*

$$|v_i(y) - U_i(y)| \leq C M_i^{-\frac{2}{n-2}}, \quad |y| \leq \delta_1 M_i^{\frac{2}{(n-2)^2}}.$$

Proof of Proposition 2.2

The proof of Proposition 2.2 consists of two steps. First we show that there exists $\delta_2(n, C_0) > 0$ small, so that

$$v_i(y) \leq C U_i(y), \quad |y| \leq \delta_2 M_i^{\frac{2}{(n-2)^2}}. \tag{2.16}$$

The proof of (2.16) is very similar to that of Lemma 3.2 in [12]. The only difference comes from the extra term $-\mu_i M_i^{-\frac{4}{n-2}} v_i$. For this we let G_1 be the Green's

function of the operator $-\Delta+\mu_i M_i^{-\frac{4}{n-2}}$ with respect to the Dirichlet condition on Ω_i (Recall that $\Omega_i := B(0, \frac{1}{10}M_i^{\frac{2}{n-2}})$). i.e.

$$\begin{cases} (-\Delta_y+\mu_i M_i^{-\frac{4}{n-2}})G_1(x,y)=\delta_x, \quad \Omega_i, \\ G_1(x,y)=0, \quad y\in\partial\Omega_i. \end{cases}$$

By direct computation

$$G_1(x,y)=\frac{1}{\omega_n(n-2)}|x-y|^{2-n}+\phi(x,y) \tag{2.17}$$

where ω_n is the area of S^{n-1}, $\phi(x,y)$ satisfies

$$|\phi(x,y)|\le \mu_i M_i^{-\frac{4}{n-2}}C(n)|x-y|^{4-n}. \tag{2.18}$$

Once we have this, the rest of the proof is very similar to that of lemma 3.2 in [12]. For $\epsilon>0$ small to be determined, there exists some constant $\delta_1\in(0,1)$, independent of i, such that for large i, let y_1 be a minimum of v_i on $|y|=\delta_1 M_i^{\frac{2}{(n-2)^2}}$, the following estimates hold:

$$\begin{aligned} v_i(y_1) &\ge \int_{\Omega_i} G_1(y_1,\eta)K_i(M_i^{-\frac{2}{n-2}}y+x_i)v_i(\eta)^{\frac{n+2}{n-2}}d\eta \\ &\ge \int_{B(0,\delta_1 M_i^{\frac{2}{(n-2)^2}})} G_1(y_1,\eta)K_i(M_i^{-\frac{2}{n-2}}y+x_i)v_i(\eta)^{\frac{n+2}{n-2}}d\eta, \end{aligned}$$

and, using (2.17) and (2.18),

$$G_1(y_1,\eta)\ge\frac{(1-\epsilon/2)}{(n-2)\omega_n}|y_1-\eta|^{2-n}\ge\frac{(1-3\epsilon/4)}{(n-2)\omega_n}|y_1|^{2-n}, \qquad |\eta|\le\delta_2|y_1|,$$

if δ_2 is chosen small enough. Now we use $\lim_{i\to\infty}K_i(x_i)=n(n-2)$ to get

$$v_i(y_1)\ge\frac{(1-\epsilon)n}{\omega_n}|y_1|^{2-n}\int_{B(0,\delta_2|y_1|)}v_i^{\frac{n+2}{n-2}}(\eta)d\eta$$

On the other hand, by Proposition 2.1

$$v_i(y_1)\le(1+\epsilon)|y_1|^{2-n}.$$

So

$$\int_{B(0,\delta_2|y_1|)}v_i^{\frac{n+2}{n-2}}(\eta)d\eta\le(1+4\epsilon)\omega_n/n.$$

A direct computation gives,
$$\int_{\mathbb{R}^n} U^{\frac{n+2}{n-2}} = \frac{\omega_n}{n}.$$
By the convergence of v_i to U, there exists some R_1, depending only on n and ϵ, such that, for large k,
$$\int_{R_1\le|\eta|\le\delta_2|y_1|} v_i^{\frac{n+2}{n-2}} d\eta \le \frac{5\epsilon}{n}\omega_n.$$
Since $v_i \le C_2$ (by(1.3))
$$\int_{R_1\le|\eta|\le\delta_2|y_1|} v_i^{\frac{2n}{n-2}} d\eta \le C_2 \int_{R_1\le|\eta|\le\delta R_k} v_i^{\frac{n+2}{n-2}} d\eta \le C\epsilon.$$

For each $2R_1 < r < \delta_2|y_1|/2$, we consider $\tilde{v}_i(z) = r^{\frac{n-2}{2}} v_i(rz)$ for $1/2 < |z| < 2$. Then $\tilde{v}_i$ satisfies
$$\Delta\tilde{v}_i(z) - \mu_i M_i^{-\frac{4}{n-2}} r^2 \tilde{v}_i(z) + K_i(M_i^{-\frac{2}{n-2}} rz + x_i)\tilde{v}_i(z)^{\frac{n+2}{n-2}} = 0, \quad 1/2 < |z| < 2.$$
We know that $\int_{\frac12\le|z|\le2} \tilde{v}_i(z)^{\frac{2n}{n-2}} \le C\epsilon$. Fix some universally small $\epsilon > 0$, we apply the Moser iteration technique to obtain $\tilde{v}_i(z) \le C$ for $\frac34 \le |z| \le \frac43$, where C is independent of k. With this, we apply the Harnack inequality to obtain $\max_{|z|=1}\tilde{v}_i(z) \le C\min_{|z|=1}\tilde{v}_i(z)$, i.e., $\max_{|y|=r} v_i(y) \le C\min_{|y|=r} v_i(y)$. Then (2.16) is established.

The second part of the proof is essentially the argument of Lemma 3.3 in [12]. We state the outline here. Let $w_i = v_i - U_i$. Here we recall that U_i satisfies
$$\begin{cases} \Delta U_i + K_i(x_i)U_i^{n^*} = 0 \quad \mathbb{R}^n, \\ \\ U_i(0) = 1 = \max_{\mathbb{R}^n} U_i. \end{cases}$$
The equation for w_i is
$$\begin{cases} (\Delta - \mu_i M_i^{-\frac{4}{n-2}} + n^* K_i(M_i^{-\frac{2}{n-2}} y + x_i)\xi_i^{\frac{4}{n-2}})w_i = E_i, \quad r \le \delta M_i^{\frac{2}{(n-2)^2}} \\ w_i(0) = 0 = |\nabla w_i(0)| \end{cases} \tag{2.19}$$
where ξ_i is obtained from the mean value theorem:
$$\xi_i^{\frac{4}{n-2}} = \int_0^1 (tv_i + (1-t)U_i)^{\frac{4}{n-2}} dt$$

and

$$E_i = \mu_i M_i^{-\frac{4}{n-2}} U_i + (K_i(x_i) - K_i(M_i^{-\frac{2}{n-2}} y + x_i)) U_i^{n^*}.$$

For E_i we clearly have

$$|E_i(y)| \le C M_i^{-\frac{4}{n-2}} (1+|y|)^{2-n} + C M_i^{-\frac{2}{n-2}} (1+|y|)^{-1-n}.$$

Let

$$\Lambda_i = \max_{B(0,\delta M_i^{\frac{2}{(n-2)^2}})} \frac{|w_i(y)|}{M_i^{-\frac{2}{n-2}}}.$$

The goal is to prove $\Lambda_i \le C$. We shall prove by contradiction. Suppose $\Lambda_i \to \infty$, let y_i be the point that λ_i is attained. Let

$$\bar{w}_i = \frac{w_i}{\Lambda_i M_i^{-\frac{2}{n-2}}}.$$

Then if $|y_i|$ is bounded, a subsequence of $\bar{w}_i$ will converge to w that satisfies

$$\begin{cases} \Delta w + n^* U^{\frac{4}{n-2}} w = 0, & \mathbb{R}^n, \\ w(0) = 0 = |\nabla w(0)|, & |w(y)| \le 1. \end{cases} \tag{2.20}$$

The only function that satisfies (2.20) is 0 (Lemma 2.4 of [12]). This violates $\bar{w}_i(y_i) = \pm 1$. This contradiction forces us to assume $y_i \to \infty$. However by the Green's representation theorem, the estimate of E_i makes it impossible to have $|\bar{w}_i(y_i)| = 1$. Proposition 2.2 is established. $\square$

2.2 Estimate of v_i over $B(0, \delta M_i^{\frac{4}{(n-2)^2}})$

Proposition 2.2 enables us to improve the estimate of the error term. In fact, from the equation for w_i we first have

$$|w_i(y)| \le C M_i^{-\frac{2}{n-2}} |y|^2, \quad |y| \le 10. \tag{2.21}$$

Consequently

$$v_i^\lambda(y) = U_i^\lambda(y) + O(M_i^{-\frac{2}{n-2}}) |y|^{-n}.$$

So we can write E_λ as (see (2.6))

$$\begin{aligned} E_\lambda \;\; = \;\; & \mu_i M_i^{-\frac{4}{n-2}} U_i^\lambda (1 - (\frac{\lambda}{r})^4) + (K_i(M_i^{-\frac{2}{n-2}} y^\lambda + x_i) - K_i(M_i^{-\frac{2}{n-2}} y + x_i))(U_i^\lambda)^{n^*} \\ & + O(M_i^{-\frac{6}{n-2}} r^{-n}) + O(M_i^{-\frac{4}{n-2}} r^{-3-n}). \end{aligned} \tag{2.22}$$

Note that the last two terms come from the difference between v_i^λ and U_i^λ. These terms will be estimated again later. Now by the Taylor expansion of K_i we have

$$E_\lambda = E_1 + E_2 + O(M_i^{-\frac{6}{n-2}} r^{1-n}) + O(M_i^{-\frac{4}{n-2}} r^{-3-n}). \tag{2.23}$$

where

$$E_1 = \mu_i M_i^{-\frac{4}{n-2}} U_i^\lambda (1 - (\frac{\lambda}{r})^4) + \frac{1}{2n} M_i^{-\frac{4}{n-2}} \Delta K_i(x_i)(\frac{\lambda^4}{r^4} - 1) r^2 (U_i^\lambda)^{n^*} \tag{2.24}$$

$$\begin{aligned} E_2 &= M_i^{-\frac{2}{n-2}} \sum_j \partial_j K_i(x_i)(\frac{\lambda^2}{r^2} - 1) r \theta_j (U_i^\lambda)^{n^*} \\ &\quad + M_i^{-\frac{4}{n-2}} \sum_{j \neq l} \partial_{jl} K_i(x_i)(\frac{\lambda^4}{r^4} - 1) r^2 \theta_j \theta_l (U_i^\lambda)^{n^*} \\ &\quad + \frac{1}{2} M_i^{-\frac{4}{n-2}} \sum_j \partial_{jj} K_i(x_i)(\frac{\lambda^4}{r^4} - 1) r^2 (\theta_j^2 - \frac{1}{n})(U_i^\lambda)^{n^*} \end{aligned}$$

where $\theta_j = y_j/r$. Note that the term of the order $M_i^{-\frac{6}{n-2}}$ has been changed due to the expansion of K_i. Each term in E_2 can be considered as a product of a radial function and an angular function. Each angular function is an eigenfunction of $-\Delta_\theta$ on S^{n-1} (θ_j corresponds to eigenvalue $n-1$, $\theta_j^2 - \frac{1}{n}$ corresponds to eigenvalue $2n$). By using the ideas in [21] [30] we construct test functions of the same form. The current purpose is to prove Proposition 2.4 in the sequel.

Before we state Proposition 2.4 we include here a proposition whose proof can be found in [21]:

Proposition 2.3. *For each $s = 1, 2$, there exists a unique C^2 radial function g_s that satisfies*

$$\begin{cases} g_s'' + \frac{n-1}{r} g_s' + (n^* K_i(x_i) \tilde{\xi}_\lambda^{\frac{4}{n-2}} - \frac{\bar{\lambda}_s}{r^2}) g_s = r^s ((\frac{\lambda}{r})^{2s} - 1)(U_i^\lambda)^{n^*}, \quad \lambda < r < M_i^{\frac{2}{n-2}}, \\ g_s(\lambda) = 0, \quad g_s(M_i^{\frac{2}{n-2}}) = 0. \end{cases}$$

where $\lambda \in (1 - \epsilon(n), 1 + \epsilon(n))$, $\epsilon(n)$ is small, $\bar{\lambda}_s = s(s + n - 2)$, $\tilde{\xi}_\lambda$ is

$$\tilde{\xi}_\lambda^{\frac{4}{n-2}} = \int_0^1 (tU_i + (1-t)U_i^\lambda)^{\frac{4}{n-2}} dt.$$

Moreover, there exists a dimensional constant $C_0(n)$ so that

$$0 \le g_s(r) \le C_0 (1 - \frac{\lambda}{r}) r^{2-n}, \quad \lambda < r < M_i^{\frac{2}{n-2}}. \tag{2.25}$$

By comparing $\tilde{\xi}_\lambda$ and ξ_λ we see that $\tilde{\xi}_\lambda$ is radial and is very close to ξ_λ for $r \le \delta_2 M_i^{\frac{2}{(n-2)^2}}$. For $r \ge \delta_2 M_i^{\frac{2}{(n-2)^2}}$ both terms are comparable to r^{2-n}.

Next we show

Proposition 2.4. *Given $\epsilon > 0$, there exists $\delta_3 > 0$ such that for all large i,*

$$\min_{|y|=r} v_i(y) \le (1+\epsilon) r^{2-n}, \quad 10 \le r \le \delta_3 M_i^{\frac{4}{(n-2)^2}}.$$

Proof of Proposition 2.4: We prove this by a contradiction. Suppose there exist $\epsilon_0 > 0$ and $\epsilon_i \to 0$ such that

$$\min_{|y|=r_i} v_i(y) \ge (1+\epsilon_0) r_i^{2-n}, \quad \text{for some } r_i \in (\delta_2 M_i^{\frac{2}{(n-2)^2}}, \epsilon_i M_i^{\frac{4}{(n-2)^2}}). \tag{2.26}$$

By the convergence of v_i to U, $r_i \to \infty$. Let $\Sigma_\lambda = B(0, \epsilon_i M_i^{\frac{4}{(n-2)^2}}) \setminus \bar{B}_\lambda$ and

$$h_1 = -M_i^{-\frac{2}{n-2}} \sum_j \partial_j K_i(x_i) \theta_j g_1(r). \tag{2.27}$$

$$h_2 = -M_i^{-\frac{4}{n-2}} g_2(r) \Big(\sum_{j \ne l} \partial_{jl} K_i(x_i) \theta_j \theta_l + \frac{1}{2} \sum_j \partial_{jj} K_i(x_i) (\theta_j^2 - \frac{1}{n}) \Big). \tag{2.28}$$

Then we have

$$(\Delta + n^* K_i(x_i) \tilde{\xi}_\lambda^{\frac{4}{n-2}})(h_1 + h_2) + E_2 = 0 \tag{2.29}$$

and, by Proposition 2.3

$$\begin{aligned} |h_1(y)| &\le C |\nabla K_i(x_i)| M_i^{-\frac{2}{n-2}} r^{2-n}, \\ |h_2(y)| &\le C M_i^{-\frac{4}{n-2}} r^{2-n}. \end{aligned} \tag{2.30}$$

Let

$$h_3(r) = Q M_i^{-\frac{4}{n-2}} f_{n,3}$$

where $Q >> 1$ is to be determined. Then first we notice that $h_3 < 0$ in Σ_λ. Also by the definition of $f_{n,\alpha}$ we have

$$\Delta h_3 = -Q M_i^{-\frac{4}{n-2}} r^{-3}, \quad \Sigma_\lambda. \tag{2.31}$$

By (2.30) and (2.15), each of $h_j, j = 1,2,3$ satisfies (2.7) and (2.8). Now by (2.29) and (2.31)

$$\begin{aligned}
& T_\lambda(h_1+h_2+h_3) \\
= & -E_2 - \mu_i M_i^{-\frac{4}{n-2}}(h_1+h_2+h_3) \\
& + n^*(K_i(M_i^{-\frac{2}{n-2}}y+x_i)\xi_\lambda^{\frac{4}{n-2}} - K_i(x_i)\tilde{\xi}_\lambda^{\frac{4}{n-2}})(h_1+h_2) \\
& - QM_i^{-\frac{4}{n-2}}r^{-3} + n^*K_i(M_i^{-\frac{2}{n-2}}y+x_i)\xi_\lambda^{\frac{4}{n-2}}h_3. \qquad (2.32)
\end{aligned}$$

Note that since $h_3 \le 0$ the last two terms have a good sign. To estimate other terms in (2.32) we first use (2.25) and (2.15) to get

$$-\mu_i M_i^{-\frac{4}{n-2}}(h_1+h_2+h_3) = O(M_i^{-\frac{6}{n-2}}r^{2-n}) + O(M_i^{-\frac{8}{n-2}}). \qquad (2.33)$$

Next by Proposition 2.2 we estimate the following:

$$\begin{aligned}
& K_i(M_i^{-\frac{2}{n-2}}y+x_i)\xi_\lambda^{\frac{4}{n-2}} - K_i(x_i)\tilde{\xi}_\lambda^{\frac{4}{n-2}} \\
= & (K_i(M_i^{-\frac{2}{n-2}}y+x_i) - K_i(x_i))\xi_\lambda^{\frac{4}{n-2}} + K_i(x_i)(\xi_\lambda^{\frac{4}{n-2}} - \tilde{\xi}_\lambda^{\frac{4}{n-2}}) \\
= & O(M_i^{-\frac{2}{n-2}}r^{-3}) + K_i(x_i)(\xi_\lambda^{\frac{4}{n-2}} - \tilde{\xi}_\lambda^{\frac{4}{n-2}}). \qquad (2.34)
\end{aligned}$$

To estimate the last term of the above, we use

$$\begin{aligned}
\xi_\lambda^{\frac{4}{n-2}} &= \int_0^1 (tv_i + (1-t)v_i^\lambda)^{\frac{4}{n-2}}dt \\
&= \int_0^1 (tU_i + (1-t)U_i^\lambda + a)^{\frac{4}{n-2}}dt \\
&= \tilde{\xi}_\lambda^{\frac{4}{n-2}} + ar^{n-6} \qquad (2.35)
\end{aligned}$$

where

$$|a| \le \begin{cases} CM_i^{-\frac{2}{n-2}}, & \lambda < r < \delta M_i^{\frac{2}{(n-2)^2}}, \\ Cr^{2-n}, & r \ge \delta M_i^{\frac{2}{(n-2)^2}}, \quad y \in O_\lambda. \end{cases} \qquad (2.36)$$

Putting (2.30) (2.34) (2.35) and (2.36) together we have

$$n^*(K_i(M_i^{-\frac{2}{n-2}}y+x_i)\xi_\lambda^{\frac{4}{n-2}} - K_i(x_i)\tilde{\xi}_\lambda^{\frac{4}{n-2}})(h_1+h_2) = O(M_i^{-\frac{4}{n-2}}r^{-3}), \quad O_\lambda. \qquad (2.37)$$

Thus, by (2.5), (2.23), (2.32), (2.33) and (2.37) we have

$$\begin{aligned} & T_\lambda(w_\lambda+h_1+h_2+h_3) \\ \leq \;& E_1+O(M_i^{-\frac{6}{n-2}}r^{2-n})+O(M_i^{-\frac{4}{n-2}}r^{-3}) \\ & +O(M_i^{-\frac{8}{n-2}})-QM_i^{-\frac{4}{n-2}}r^{-3}, \quad \text{in} \quad O_\lambda. \end{aligned}$$

It is easy to verify that

$$|E_1(y)|\leq CM_i^{-\frac{4}{n-2}}r^{2-n}, \quad M_i^{-\frac{8}{n-2}}=\circ(1)M_i^{-\frac{4}{n-2}}r^{-3}.$$

So by choosing Q large enough we have

$$T_\lambda(w_\lambda+h_1+h_2+h_3)\leq 0 \quad O_\lambda.$$

Proposition 2.4 is established.

Remark 2.2. *In the proof of Proposition 2.4 we don't need the sign of* $\Delta K_i(x_i)$*.*

2.3 The vanishing rate of $|\nabla K_i(x_i)|$

Next we improve the estimate of v_i-U_i: First we have

Proposition 2.5. *There exist* $\delta_4>0$ *and* $C>0$ *such that*

$$v_i(y)\leq CU_i(y) \quad |y|\leq \delta_4 M_i^{\frac{4}{(n-2)^2}}.$$

The proof is similar to the step one of Proposition 2.2. $\square$

To further estimate v_i-U_i more precisely, we need the following Pohozaev Identity for

$$\Delta f(x)-tf(x)+H(x)f(x)^{n^*}=0 \quad B_\sigma.$$

$$\begin{aligned} & \int_{B_\sigma}(\frac{n-2}{2n}(\nabla H\cdot x)f^{\frac{2n}{n-2}}-tf^2) \\ = & \int_{\partial B_\sigma}(\frac{n-2}{2n}\sigma H f^{\frac{2n}{n-2}}+\sigma|\frac{\partial f}{\partial\nu}|^2-\frac{\sigma}{2}|\nabla f|^2+\frac{n-2}{2}\frac{\partial f}{\partial\nu}f-\frac{t}{2}\sigma f^2). \end{aligned}$$

Let $\tilde v_i(y)=v_i(y+e)$, where $e=\frac{\nabla K_i(x_i)}{|\nabla K_i(x_i)|}$ is a unit vector. Let

$$\tilde K_i(y)=K_i(M_i^{-\frac{2}{n-2}}(y+e)+x_i),$$

then $\tilde{v}_i$ satisfies

$$\Delta \tilde{v}_i(y) - \mu_i M_i^{-\frac{4}{n-2}} \tilde{v}_i(y) + \tilde{K}_i(y)\tilde{v}_i(y)^{n^*} = 0, \quad |y| \le L_i := \frac{\delta_4}{2} M_i^{\frac{4}{(n-2)^2}}.$$

So the Pohozaev Identity applied to $\tilde{v}_i$ over B_{L_i} gives

$$\begin{aligned}
&\int_{B_{L_i}} \left(\frac{n-2}{2n} (\nabla \tilde{K}_i(y) \cdot y) \tilde{v}_i^{\frac{2n}{n-2}} - \mu_i M_i^{-\frac{4}{n-2}} \tilde{v}_i^2(y) \right) dy \\
= &\int_{\partial B_{L_i}} (\frac{n-2}{2n} L_i \tilde{K}_i(y) \tilde{v}_i^{\frac{2n}{n-2}} + L_i |\frac{\partial \tilde{v}_i}{\partial \nu}|^2 - \frac{L_i}{2} |\nabla \tilde{v}_i|^2 \\
&+ \frac{n-2}{2} \frac{\partial \tilde{v}_i}{\partial \nu} \tilde{v}_i - \frac{\mu_i}{2} M_i^{-\frac{4}{n-2}} L_i \tilde{v}_i^2). \qquad (2.38)
\end{aligned}$$

By Proposition 2.5 and standard elliptic estimates the right hand side of (2.38) is $O(M_i^{-\frac{4}{n-2}})$. Then by using $e = \nabla K_i(x_i)/|\nabla K_i(x_i)|$ we see that the left hand side of the Pohozaev identity is greater than

$$C|\nabla K_i(x_i)| M_i^{-\frac{2}{n-2}} + O(M_i^{-\frac{4}{n-2}})$$

for some $C > 0$. Consequently

$$|\nabla K_i(x_i)| \le C M_i^{-\frac{2}{n-2}}. \qquad (2.39)$$

Base on (2.39) we can write the equation for $w_i := v_i - U_i$ as

$$(\Delta + n^* K_i(M_i^{-\frac{2}{n-2}} y + x_i) \xi_i^{\frac{4}{n-2}}) w_i = O(M_i^{-\frac{4}{n-2}})(1+r)^{2-n}, \quad r \le \delta_4 M_i^{\frac{4}{(n-2)^2}}.$$

Then the same estimate in Proposition 2.2 gives

$$|v_i(y) - U_i(y)| \le C M_i^{-\frac{4}{n-2}}, \quad |y| \le \delta_4 M_i^{\frac{4}{(n-2)^2}}. \qquad (2.40)$$

Using the fact that $v_i(0) = U_i(0)$ and $\nabla v_i(0) = \nabla U_i(0) = 0$, we have

$$|v_i(y) - U_i(y)| \le C M_i^{-\frac{4}{n-2}} |y|^2, \quad |y| \le 10. \qquad (2.41)$$

2.4 Harnack inequality on $B(0, \delta M_i^{\frac{2}{n-2}})$

Now we establish

Proposition 2.6. *For any $\epsilon > 0$, there exists $\delta_5 > 0$ depending on ϵ and n such that*

$$\min_{|y|=r} v_i(y) \leq (1+\epsilon) r^{2-n}, \quad r \leq \delta_5 M_i^{\frac{2}{n-2}}.$$

Proof of Proposition 2.6: We still prove it by a contradiction by assuming that there exist ϵ_0 and $\epsilon_i \to 0$ such that

$$\min_{|y|=\epsilon_i M_i^{\frac{2}{n-2}}} v_i(y) \geq (1+\epsilon_0)|y|^{2-n}. \tag{2.42}$$

As a consequence of (2.41) (see also (2.21))

$$v_i^\lambda(y) = U_i^\lambda(y) + O(M_i^{-\frac{4}{n-2}}|y|^{-n}).$$

Therefore, in stead of (2.23) we now have

$$E_\lambda = E_1 + E_2 + O(M_i^{-\frac{8}{n-2}} r^{1-n})(1 - \frac{\lambda}{r}) + O(M_i^{-\frac{6}{n-2}} r^{-3-n})(1 - \frac{\lambda}{r}). \tag{2.43}$$

Note that we include $1 - \frac{\lambda}{r}$ deliberately because the dominant term now vanishes on ∂B_λ. We still construct h_1 and h_2 as in (2.27) and (2.28). Because of the new rate of $|\nabla K_i(x_i)|$ we now have

$$|h_1| + |h_2| = O(M_i^{-\frac{4}{n-2}} r^{2-n}).$$

We note that (2.29) also holds. Now we have

$$\begin{aligned} T_\lambda(h_1 + h_2) &= -E_2 - \mu_i M_i^{-\frac{4}{n-2}}(h_1 + h_2) \\ &\quad + n^*(K_i(M_i^{-\frac{2}{n-2}} y + x_i)\xi_\lambda^{\frac{4}{n-2}} - K_i(x_i)\tilde{\xi}_\lambda^{\frac{4}{n-2}})(h_1 + h_2). \end{aligned}$$

By the new rate of $|\nabla K_i(x_i)|$ we have

$$|\mu_i M_i^{-\frac{4}{n-2}}(h_1 + h_2)| \leq C M_i^{-\frac{8}{n-2}} r^{2-n}(1 - \frac{\lambda}{r}).$$

Also by using this new rate of $|\nabla K_i(x_i)|$ in (2.34) we have

$$\begin{aligned} & K_i(M_i^{-\frac{2}{n-2}}y+x_i)\xi_\lambda^{\frac{4}{n-2}}-K_i(x_i)\tilde{\xi}_\lambda^{\frac{4}{n-2}} \\ = \;& O(M_i^{-\frac{4}{n-2}}r^{-2})+K_i(x_i)(\xi_\lambda^{\frac{4}{n-2}}-\tilde{\xi}_\lambda^{\frac{4}{n-2}}). \end{aligned}$$

Corresponding to (2.35) and (2.36) we now have

$$\xi_\lambda^{\frac{4}{n-2}}=\tilde{\xi}_\lambda^{\frac{4}{n-2}}+ar^{n-6}\quad O_\lambda$$

where

$$|a|\le\begin{cases} O(M_i^{-\frac{4}{n-2}}), & \lambda<r<\delta M_i^{\frac{4}{(n-2)^2}}, \\ Cr^{2-n}, & \delta M_i^{\frac{4}{(n-2)^2}}\le r\le\epsilon_i M_i^{\frac{2}{n-2}}. \end{cases}$$

Consequently

$$\begin{aligned} & n^*(K_i(M_i^{-\frac{2}{n-2}}y+x_i)\xi_\lambda^{\frac{4}{n-2}}-K_i(x_i)\tilde{\xi}_\lambda^{\frac{4}{n-2}})(h_1+h_2) \\ = \;& O(M_i^{-\frac{8}{n-2}}r^{2-n})(1-\frac{\lambda}{r})+\tilde{E}_3. \end{aligned}$$

where $\tilde{E}_3$ is 0 when $r\le\delta_4 M_i^{\frac{4}{(n-2)^2}}$ and is $O(M_i^{-\frac{4}{n-2}}r^{-2-n})$ when $r\ge\delta_4 M_i^{\frac{4}{(n-2)^2}}$.

Now we can combine w_λ with h_1,h_2:

$$\begin{aligned} & T_\lambda(w_\lambda+h_1+h_2) \\ \le\;& E_1+O(M_i^{-\frac{8}{n-2}}r^{2-n})(1-\frac{\lambda}{r})+O(M_i^{-\frac{6}{n-2}}r^{-3-n})(1-\frac{\lambda}{r})+\tilde{E}_3. \end{aligned}$$

Now the second term in E_1 becomes the dominant term. In fact, since $\Delta K_i(x_i)$ is large we have

$$\begin{aligned} & T_\lambda(w_\lambda+h_1+h_2) \\ \le\;& \mu_i M_i^{-\frac{4}{n-2}}U_i^\lambda(1-(\frac{\lambda}{r})^4)-\frac{1}{3n}M_i^{-\frac{4}{n-2}}\Delta K_i(x_i)(1-(\frac{\lambda}{r})^4)r^2(U_i^\lambda)^{n^*}. \end{aligned}\tag{2.44}$$

Note that $\Delta K_i(x_i)$ makes the right hand side of (2.44) negative when r is close to, or comparable to λ. But the right hand side becomes positive when r is large. So we need to construct the following function to deal with this. Let

$$h_3(y)=\int_{\Sigma_\lambda}\mu_i M_i^{-\frac{4}{n-2}}G(y,\eta)U_i^\lambda(\eta)(1-(\frac{\lambda}{|\eta|})^4)d\eta.$$

For h_3 we have

$$0 \le h_3(y) \le C(C_0, n) M_i^{-\frac{4}{n-2}} |y|^{4-n},$$

(so $h_3 = \circ(1)|y|^{2-n}$) and

$$-\Delta h_3 = \mu_i M_i^{-\frac{4}{n-2}} U_i^\lambda (1 - (\frac{\lambda}{|\eta|})^4), \quad \Sigma_\lambda.$$

Because of this equation, the bad term now becomes $n^* K_i(M_i^{-\frac{2}{n-2}} y + x_i)\xi_\lambda^{\frac{4}{n-2}} h_3$. To be more precise, we have

$$\begin{aligned}
& T_\lambda(w_\lambda + \sum_{s=1}^{3} h_s) \\
\le\ & -\frac{1}{3n} M_i^{-\frac{4}{n-2}} \Delta K_i(x_i)(1 - (\frac{\lambda}{r})^4) r^2 (U_i^\lambda)^{n^*} + n^* K_i(M_i^{-\frac{2}{n-2}} y + x_i)\xi_\lambda^{\frac{4}{n-2}} h_3 \\
\le\ & -\frac{1}{3n} M_i^{-\frac{4}{n-2}} \Delta K_i(x_i)(1 - (\frac{\lambda}{r})^4) r^2 (U_i^\lambda)^{n^*} \\
& + C(n, C_0) M_i^{-\frac{4}{n-2}} |y|^{4-n} (1 + |y|)^{-4} (1 - \lambda/r) \\
\le\ & 0
\end{aligned}$$

where in the last step we used the largeness of $\Delta K_i(x_i)$. With this inequality the moving sphere argument applies as before to get a contradiction. Proposition 2.6 is established. □

2.5 The completion of the proof of Theorem 1.1

By Proposition 2.6 and the step one of the proof of Proposition 2.2 we have, for some $\delta_5 > 0$ small and C large

$$v_i(y) \le C U_i(y), \quad |y| \le \delta_5 M_i^{\frac{2}{n-2}}. \tag{2.45}$$

Note that we can not get better estimate on $v_i - U_i$ as before because terms of order $O(M_i^{-\frac{4}{n-2}})$ prevent us from getting estimates better than (2.40) and (2.41). By using the vanishing rate of $|\nabla K_i(x_i)|$ (2.39) and (2.45), (2.40),(2.41) we shall get a contradiction to (2.1) from the Pohozaev identity as follows.

Let $L_i = \delta_5 M_i^{\frac{2}{n-2}}$, we apply the Pohozaev identity to v_i on B_{L_i}. Then the right hand side is

$$\int_{\partial B_{L_i}} \{\frac{n-2}{2n} L_i K_i v_i^{\frac{2n}{n-2}} + L_i |\frac{\partial v_i}{\partial \nu}|^2 - \frac{L_i}{2} |\nabla v_i|^2 + \frac{n-2}{2} \frac{\partial v_i}{\partial \nu} v_i - \frac{\mu_i M_i^{-\frac{4}{n-2}}}{2} L_i v_i^2\}.$$

By (2.45) and the corresponding gradient estimate the right hand side of the Pohozaev identity is $O(M_i^{-2})$.

The left hand side of the Pohozaev identity is

$$\int_{B_{L_i}} \{\frac{n-2}{2n} M_i^{-\frac{2}{n-2}} (\nabla K_i(M_i^{-\frac{2}{n-2}} y + x_i) \cdot y) v_i^{\frac{2n}{n-2}} - \mu_i M_i^{-\frac{4}{n-2}} v_i^2\}$$

We call the first term of the above L_1, the second term L_2. Clearly by the convergence of v_i to U, we have

$$L_2 \geq -\mu_i M_i^{-\frac{4}{n-2}} C(n).$$

Now we estimate L_1, for which we first have

$$\nabla K_i(M_i^{-\frac{2}{n-2}} y + x_i) \cdot y = \sum_j \partial_j K_i(x_i) y_j + M_i^{-\frac{2}{n-2}} \sum_{l,j} \partial_{jl} K_i(x_i) y_j y_l + O(M_i^{-\frac{4}{n-2}} r^3).$$

Then we can write L_1 as

$$\begin{aligned} L_1 &= \frac{n-2}{2n} M_i^{-\frac{2}{n-2}} \int_{B_{L_i}} \sum_j \partial_j K_i(x_i) y_j v_i^{\frac{2n}{n-2}} \\ &\quad + \frac{n-2}{2n} M_i^{-\frac{4}{n-2}} \int_{B_{L_i}} \sum_{jl} \partial_{jl} K_i(x_i) y_j y_l v_i^{\frac{2n}{n-2}} dy + O(M_i^{-\frac{6}{n-2}}) \\ &= L_{11} + L_{12} + O(M_i^{-\frac{6}{n-2}}). \end{aligned}$$

To estimate L_{11} let $\bar{L}_i = M_i^{\frac{1}{n-2}}$, then

$$\begin{aligned} \int_{B_{L_i}} y_j v_i^{\frac{2n}{n-2}} &= \int_{B_{\bar{L}_i}} + \int_{B_{L_i} \setminus B_{\bar{L}_i}} \\ &= \int_{B_{\bar{L}_i}} y_j (U_i + O(M_i^{-\frac{4}{n-2}}))^{\frac{2n}{n-2}} dy + O(M_i^{-1}) \\ &= O(M_i^{-\frac{4}{n-2}}) + O(M_i^{-1}). \end{aligned}$$

So by (2.39) $L_{11} = \circ(M_i^{-\frac{4}{n-2}})$. By similar estimate we see that the leading term in L_{12} is $\frac{n-2}{2n^2} M_i^{-\frac{4}{n-2}} \Delta K_i(x_i) \int_{\mathbb{R}^n} |y|^2 U_i^{\frac{2n}{n-2}} dy$. Then the largeness of $\Delta K_i(x_i)$ clearly leads to a contradiction in the Pohozaev identity. Theorem 1.1 is established. $\Box$

2.6 The Proof of Corollary 1.1

Let

$$u(y) = (\frac{2}{1+|y|^2})^{\frac{n-2}{2}} w(\pi^{-1}(y)), \quad y \in \mathbb{R}^n$$

and g_0 denote the standard metric on $\mathbb{S}^n$. In stereographic projection

$$g_0 = \sum_{i=1}^{n+1} dx_i^2 = \left\{ (\frac{2}{1+|y|^2})^{\frac{n-2}{2}} \right\}^{\frac{4}{n-2}} dy^2.$$

Since the north pole is not a critical point of R, we know $u \sim O(|y|^{2-n})$ at infinity. Then the equation for u becomes

$$\begin{cases} \Delta u(y) - \mu(y)(\frac{2}{1+|y|^2})^2 u(y) + K(y) u(y)^{\frac{n+2}{n-2}} = 0, \quad \mathbb{R}^n, \\ u(y) \sim O(|y|^{2-n}) \quad \text{at } \infty. \end{cases} \tag{2.46}$$

By applying Theorem 1.1 we obtain

$$u(y) \leq C, \quad y \in \mathbb{R}^n. \tag{2.47}$$

If $\mu \equiv 0$, the result in [30] yields (2.47) only under the assumption $\Delta K(y) > 0$ for each critical point y. The upper bound on u gives the upper bound on w, then by Harnack inequality

$$\frac{1}{C} \leq w(x) \leq C, \quad \text{on} \quad \mathbb{S}^n.$$

Corollary 1.1 is established. □

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